ON WARING'S PROBLEM: A SQUARE, FOUR CUBES AND A BIQUADRATE

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1. Introduction. Additive representations of natural numbers by mixtures of squares, cubes and biquadrates belong to the class of more interesting special cases which form the object of attention for testing the general expectation that any sufficiently large natural number n is representable in the form

$$x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s} = n,$$

as soon as the reciprocal sum $\sum_{j=1}^{s} k_j^{-1}$ is reasonably large. With the exception of a handful of very special problems, in the current state of knowledge the latter reciprocal sum must exceed 2, at the very least, in order that it be feasible to successfully apply the Hardy-Littlewood method to treat the corresponding additive problem. Here we remove a case from the list of those combinations of exponents which have defied treatment thus far.

Theorem. *Let $\nu(n)$ denote the number of representations of the natural number n as the sum of a square, four cubes and a biquadrate. Then $\nu(n) \gg n^{13/12}$.*

We remark that the lower bound for $\nu(n)$ provided by our theorem is of the same order of magnitude as the main term of the conjectured asymptotic formula for $\nu(n)$ predicted by a formal application of the circle method. Our result, as well as the method of proof, should be compared with the work of Vaughan [11], who obtained a theorem of similar strength for the sum of one square and five cubes. Although our proof has many features in common with Vaughan's treatment, it should be stressed that the problem under consideration here seems to require tools which became available only very recently. In this context we direct the reader's attention to the use of "breaking classical convexity" (see Wooley [15]), which supplies a good bound for the fifth moment of a cubic smooth Weyl sum, and a refined treatment of a classical cubic Weyl sum restricted to excessively large major arcs (see Brüdern [3]). It would appear to be difficult to establish our theorem without these two tools. In this context we remark that, subject to the truth of

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an unproved hypothesis concerning certain Hasse-Weil L-functions, one has sharp estimates for the sixth moment of the cubic Weyl sum due to Hooley [8] and Heath-Brown [6], and these would permit the proof of a conditional asymptotic formula for $\nu(n)$.

For other results on mixed sums of squares, cubes and fourth powers we refer the reader to Hooley [7], Brüdern [1], Kawada and Wooley [9] and Brüdern, Kawada and Wooley [4].

Throughout, ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, with implicit constants depending at most on ε. Throughout, the letter p will denote a prime number. In an effort to simplify our analysis, we adopt the following convention concerning the number ε. Whenever ε appears in a statement, either implicitly or explicitly, we assert that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε.

2. Preliminaries. We first set the scene with some notation. Let n be a large natural number, and write

$$P_k = n^{1/k} \quad (k = 2, 3, 4), \quad M = P_4^{1/9} \quad \text{and} \quad L = (\log n)^{1/100}. \tag{1}$$

We define the set of R-smooth numbers up to P by

$$\mathcal{A}^*(P, R) = \{n \in [P/2, P] \cap \mathbb{Z} : p|n \text{ implies } p \le R\}.$$

Let η be a small positive number to be chosen later, and define the exponential sums

$$f(\alpha) = \sum_{1 \le x \le P_2} e(\alpha x^2), \quad g(\alpha) = \sum_{y \in \mathcal{A}^*(P_3, P_3^\eta)} e(\alpha y^3)$$

and

$$h(\alpha) = \sum_{M < p \le M^{1+\eta}} \sum_{1 \le z \le P_4/p} e(\alpha (pz)^4). \tag{2}$$

Also, we require the weighted exponential sum

$$G(\alpha) = \sum_{1 \le x \le 2P_3} \Gamma(x/P_3) e(\alpha x^3),$$

where

$$\Gamma(t) = \exp(-1/(1 - (t-1)^2)).$$

Whenever it is convenient so to do, and confusion is easily avoided, we omit mention of the parameter α from the exponential sums $f(\alpha)$, $G(\alpha)$, $g(\alpha)$ and $h(\alpha)$.

Finally, we define the integral

$$\nu_0(n) = \int_0^1 f(\alpha) G(\alpha) g(\alpha)^3 h(\alpha) e(-\alpha n) d\alpha, \tag{3}$$

and observe that one has

$$\nu(n) \gg \nu_0(n). \tag{4}$$

3. Auxiliary mean value estimates. Before launching our application of the Hardy-Littlewood method, it is convenient to record the various mean value estimates which have prompted our recondite choice of generating functions.

First we recall from the proof of Theorem 1.2 of Wooley [15] that whenever η is a sufficiently small positive number, as we henceforth assume, one has

$$\int_0^1 |g(\alpha)|^5 d\alpha \ll P_3^{44/17}. \tag{5}$$

Next we record the simple bound

$$\int_0^1 |G(\alpha)|^4 d\alpha \ll P_3^{2+\varepsilon}, \tag{6}$$

which follows on interpreting the integral as a weighted sum over the solutions of the equation $x_1^3 + x_2^3 = x_3^3 + x_4^3$, with $1 \le x_i \le 2P_3$ $(1 \le i \le 4)$, and applying Hua's Lemma (see, for example, Lemma 2.5 of Vaughan [13]).

Finally, we consider the mean value

$$V = \int_0^1 |g(\alpha)^2 h(\alpha)^4| d\alpha. \tag{7}$$

On applying Hölder's inequality to the exponential sum (2), one obtains

$$|h(\alpha)|^4 \le M^{3(1+\eta)} \sum_{M<p\le M^{1+\eta}} \Big| \sum_{1\le z\le P_4/p} e(\alpha(pz)^4) \Big|^4.$$

Substituting the latter inequality into (7), it follows from a consideration of the underlying diophantine system that $V \le M^{3(1+\eta)} V_0$, where V_0 denotes the number of integral solutions of the equation

$$x_1^3 - x_2^3 = p^4(y_1^4 + y_2^4 - y_3^4 - y_4^4) \tag{8}$$

subject to

$$x_i \in \mathcal{A}^*(P_3, P_3^\eta)\ (i = 1, 2), \quad M < p \le M^{1+\eta} \quad \text{and} \quad 1 \le y_j \le P_4/p\ (1 \le j \le 4).$$

Since $P_3^\eta < M$, we find that in any solution $\mathbf{x}, \mathbf{y}, p$ of (8) counted by V_0, one has $p \nmid x_1 x_2$. Consequently, on dividing up the range for p into dyadic intervals, the bound $V_0 \ll P_3^{1+\varepsilon} M^{1+\eta} (P_4/M)^2$ is an immediate consequence of Lemma 1 of Brüdern [2]. We therefore infer that

$$V \ll P_3^{1+\varepsilon} P_4^2 M^{2+4\eta}. \tag{9}$$

4. Initial pruning. We define our Hardy-Littlewood dissection of the unit interval as follows. When $1 \le X \le \frac{1}{2}P_2$, we define the major arcs $\mathfrak{M}(X)$ to be the union of the intervals

$$\mathfrak{M}(q,a;X) = \{\alpha \in [0,1) \ : \ |q\alpha - a| \le Xn^{-1}\},$$

with $0 \le a \le q \le X$ and $(a,q)=1$. In the interests of brevity, we then write

$$\mathfrak{M} = \mathfrak{M}(n^{10/21}), \quad \mathfrak{N} = \mathfrak{M}(n^{13/42}), \quad \mathfrak{K} = \mathfrak{M}(L).$$

Finally, we define the minor arcs $\mathfrak{m}$ by $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$.

The contribution of the minor arcs to $\nu_0(n)$ is easily bounded by applying the mean value estimates of the previous section together with Weyl's inequality. For the latter inequality (see Lemma 2.4 of Vaughan [13]) yields the estimate

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P_2^{11/21+\varepsilon},$$

whence, on recalling (5)-(7) and (9), it follows from Hölder's inequality that

$$\begin{aligned}\int_{\mathfrak{m}} |fGg^3h| d\alpha &\ll P_2^{11/21+\varepsilon} V^{1/4} \Big(\int_0^1 |g|^5 d\alpha \Big)^{1/2} \Big(\int_0^1 |G|^4 d\alpha \Big)^{1/4} \\ &\ll P_2^{11/21+2\varepsilon} \Big(P_3 P_4^2 M^{2+4\eta} \Big)^{1/4} (P_3^{44/17})^{1/2} (P_3^2)^{1/4}.\end{aligned}$$

In view of (1), a modicum of computation therefore reveals that

$$\int_{\mathfrak{m}} |f(\alpha)G(\alpha)g(\alpha)^3h(\alpha)| d\alpha \ll n^{\frac{13}{12}-\eta}. \tag{10}$$

In order to prune the major arcs $\mathfrak{M}$ back to the narrower set of arcs $\mathfrak{N}$, we make use of Theorem 2 of Brüdern [3], which establishes that for $1 \le X \le P_3^{10/7}$, one has

$$\int_{\mathfrak{M}(X)} |G(\alpha)|^4 d\alpha \ll n^{\varepsilon}(X^{7/2}P_3^{-3} + X^2P_3^{-1} + P_3).$$

For each X with $1 \le X \le \frac{1}{2}P_2$, write $\mathfrak{M}^*(X) = \mathfrak{M}(X) \setminus \mathfrak{M}(\frac{1}{2}X)$. Then since Weyl's inequality supplies the bound

$$\sup_{\alpha \in \mathfrak{M}^*(X)} |f(\alpha)| \ll P_2^{1+\varepsilon} X^{-1/2},$$

it follows that for $n^{13/42} \le X \le n^{10/21}$, one has

$$\int_{\mathfrak{M}^*(X)} |f(\alpha)G(\alpha)|^4 d\alpha \ll n^{\varepsilon}(X^{3/2}n + n^{5/3} + n^{7/3}X^{-2}) \ll n^{12/7+\varepsilon}.$$

A dyadic dissection therefore shows that

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |f(\alpha)G(\alpha)|^4 d\alpha \ll n^{12/7+\varepsilon},$$

whence by applying Hölder's inequality and recalling (5)-(7) and (9), we obtain

$$\int_{\mathfrak{M}\setminus\mathfrak{N}} |fGg^3h| d\alpha \ll V^{1/4}\Big(\int_0^1 |g|^5 d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{M}\setminus\mathfrak{N}} |fG|^4 d\alpha\Big)^{1/4} \ll n^{\frac{13}{12}-\eta}.$$

On recalling (10), (3) and (4) we may conclude thus far that

$$\nu(n) \gg \int_{\mathfrak{N}} f(\alpha)G(\alpha)g(\alpha)^3 h(\alpha)e(-\alpha n) d\alpha + O(n^{13/12-\eta}). \tag{11}$$

5. Further pruning. The next step in the pruning argument is more routine and can be handled in many ways. We take a route economical in terms of space, though not so economical in its use of the literature. We note first that the number of representations of an integer n in the form $n = pz$, with $M < p \le M^{1+\eta}$ and $1 \le z \le P_4/p$, is at most $O(1)$. Consequently, the argument of the proof of Théorème $2'$(i) of Tenenbaum [10] (see §2, and in particular the estimation of W on p.235) provides the estimate

$$\int_0^1 |f(\alpha)^2 h(\alpha)^4| d\alpha \ll nL^{\varepsilon}. \tag{12}$$

Next observe that a standard application of the Hardy-Littlewood method (see, for example, §2 of Vaughan [13]) shows that whenever $t > 4$ is real, one has

$$\int_0^1 |f(\alpha)|^t d\alpha \ll P_2^{t-2}. \tag{13}$$

Moreover the work of §2 of Brüdern [3], in combination with the methods of §4.4 of Vaughan [13], reveals that whenever X is a real number with $1 \le X \le P_3^{1/2}$ and $t > 4$, then one has

$$\int_{\mathfrak{N}\setminus\mathfrak{M}(X)} |G(\alpha)|^t d\alpha \ll P_3^{t-3} X^{\varepsilon-(t-4)/3}. \tag{14}$$

We remark that in our application of the latter estimate below, we take $X = L$. Finally, as a consequence of Theorem 2 of Brüdern and Wooley [5], it follows that whenever $t > \frac{77}{10}$ one has

$$\int_0^1 |g(\alpha)|^t d\alpha \ll P^{t-3}. \tag{15}$$

An application of Hölder's inequality now yields

$$\int_{\mathfrak{N}\setminus\mathfrak{K}} |fGg^3h|d\alpha$$
$$\ll \Big(\int_0^1 |f^2h^4|d\alpha\Big)^{\frac14}\Big(\int_0^1 |g|^{\frac{164}{21}}d\alpha\Big)^{\frac{63}{164}}\Big(\int_{\mathfrak{N}\setminus\mathfrak{K}} |G|^{\frac{41}{10}}d\alpha\Big)^{\frac{10}{41}}\Big(\int_0^1 |f|^{\frac{41}{10}}d\alpha\Big)^{\frac{5}{41}},$$

so that on making use of the estimates (12)-(15), we deduce from (11) that

$$\nu(n) \gg \int_{\mathfrak{K}} f(\alpha)G(\alpha)g(\alpha)^3h(\alpha)e(-\alpha n)d\alpha + O(n^{13/12}L^{-\eta}). \qquad (16)$$

6. The denouement. The major arcs $\mathfrak{K}$ are extremely narrow, and thus it is essentially a routine matter to replace the generating functions f, G, g and h by their respective standard major arc approximants. When $k = 2, 3$ or 4, write

$$S_k(q,a) = \sum_{r=1}^{q} e(ar^k/q)$$

and write also

$$v_k(\beta) = \int_0^{P_k} e(\beta\gamma^k)d\gamma \quad (k = 2,4), \quad v_3(\beta) = \int_{P_3/2}^{P_3} e(\beta\gamma^3)d\gamma.$$

Further, when $\alpha \in \mathfrak{M}(q,a;L) \subseteq \mathfrak{K}$, define

$$w_k(\alpha) = q^{-1}S_k(q,a)v_k(\alpha - a/q).$$

Then by Theorem 4.1 of Vaughan [13], one has

$$\sup_{\alpha\in\mathfrak{K}} |f(\alpha) - w_2(\alpha)| \ll L.$$

Also, it follows from Lemma 8.5 of Wooley [14] (see also Lemma 5.4 of Vaughan [12] for a related conclusion) that there exists a positive number c, depending only on η, such that

$$\sup_{\alpha\in\mathfrak{K}} |g(\alpha) - cw_3(\alpha)| \ll P_3L^{-10}.$$

Suppose next that $\alpha \in \mathfrak{M}(q,a;L) \subseteq \mathfrak{K}$. Then on recalling (2), it is apparent from Theorem 4.1 of Vaughan [13] that

$$h(\alpha) = \sum_{M<p\le M^{1+\eta}} \Big(q^{-1}S_4(q,ap^4)\int_0^{P_4/p} e(p^4\gamma^4(\alpha - a/q))d\gamma + O(q^{1/2+\varepsilon})\Big).$$

However, the condition $p > M$ occurring in the summation, together with the implicit hypothesis that $q \le L < M$, ensures that $p \nmid q$, and so by a change of variable one finds that for each prime p occurring in the latter summation, one has $S_4(q,ap^4)=S_4(q,a)$. Thus, following an obvious change of variable, we arrive at the conclusion

$$\sup_{\alpha\in\mathfrak{K}}|h(\alpha)-\Xi w_4(\alpha)|\ll M^{1+2\eta},$$

where

$$\Xi=\sum_{M<p\le M^{1+\eta}}p^{-1}.$$

We note for future reference that elementary prime number theory supplies the estimate $\Xi=\log(1+\eta)+O(L^{-10})$.

Finally, on writing

$$V(\beta)=\int_0^{2P_3}\Gamma(\gamma/P_3)e(\beta\gamma^3)d\gamma,$$

and defining the function $W(\alpha)$ for $\alpha\in\mathfrak{M}(q,a;L)\subseteq\mathfrak{K}$ by

$$W(\alpha)=q^{-1}S_3(q,a)V(\alpha-a/q),$$

we find that Lemma 2 of Brüdern [3] supplies the bound

$$\sup_{\alpha\in\mathfrak{K}}|G(\alpha)-W(\alpha)|\ll 1.$$

Collecting together the above estimates, and writing

$$T(q,a)=q^{-6}S_2(q,a)S_3(q,a)^4S_4(q,a)$$

and

$$u(\beta)=v_2(\beta)V(\beta)v_3(\beta)^3v_4(\beta),$$

we deduce that when $\alpha\in\mathfrak{M}(q,a;L)\subseteq\mathfrak{K}$, one has

$$|f(\alpha)G(\alpha)g(\alpha)^3h(\alpha)-\Xi c^3T(q,a)u(\alpha-a/q)|\ll P_2P_3^4P_4L^{-10}.$$

Since the measure of $\mathfrak{K}$ is $O(L^2n^{-1})$, it follows that

$$\begin{aligned}\int_{\mathfrak{K}}&f(\alpha)G(\alpha)g(\alpha)^3h(\alpha)e(-\alpha n)d\alpha\\ &=\Xi c^3\sum_{1\le q\le L}\sum_{\substack{a=1\\(a,q)=1}}^{q}T(q,a)e(-an/q)\int_{-L/(qn)}^{L/(qn)}u(\beta)e(-\beta n)d\beta\\ &\quad+O(n^{13/12}L^{-1}).\end{aligned}\tag{17}$$

The bounds

$$T(q,a)\ll q^{\varepsilon-25/12}\quad\text{and}\quad u(\beta)\ll n^{25/12}(1+n|\beta|)^{-25/12}$$

are immediate from Lemma 2.8 and Theorem 4.2 of Vaughan [13], together with Lemma 7 of Brüdern [3]. Thus a routine argument permits the replacement of the integral in (17) by the singular integral

$$I(n) = \int_{-\infty}^{\infty} u(\beta)e(-\beta n)d\beta,$$

and also allows the completion of the sum in (17) to the singular series

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} T(q,a)e(-an/q),$$

with acceptable errors which contribute at most $O(n^{13/12}L^{-1/20})$ within (17). Moreover, standard endgame technique from the theory of the Hardy-Littlewood method, which we omit here in the interest of saving space, demonstrates with ease that $\mathfrak{S}(n) \gg 1$ and $I(n) \gg n^{13/12}$. Thus, on recalling (16), we arrive at the lower bound

$$\nu(n) \gg \Xi c^3 \mathfrak{S}(n) I(n) + O(n^{13/12}L^{-\eta}) \gg n^{13/12},$$

and the proof of our theorem is complete.

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